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THREE-DIMENSIONAL SUB- AND SUPERSONIC FLOWS IN NOZZLES AND CHANNELS OF VARYING CROSS SECTION

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Three-dimensional sub- and supersonic flows of gas in nozzles and channels of varying cross section are analyzed. The inverse problem of the theory of Laval nozzles is formulated and extended to three-dimensional flows. An implicit three-point difference scheme with varying pitch along a layer is proposed. In the neighborhood of the surface for which the Cauchy data are specified an asymptotic series expansion in terms of the stream-function is derived and the method of solving related equations is indicated. Examples of calculations of three-dimensional flows in nozzles are presented. Papers [1 - 3] dealing with three-dimensional supersonic flows in nozzles and paper [4] in which an analytical solution is derived for the flow in the neighborhood of the nozzle center should be noted among recent publications.

1. Fundamental equations and statement of problem. We introduce a system of curvilinear coordinates linked with the curve $y = f_0(s)$ lying in the xy-plane. The coordinates of a point are defined in this system by the arc length s, the distance r along the normal to this curve, and by the angle φ in a plane normal to it (Fig. 1).

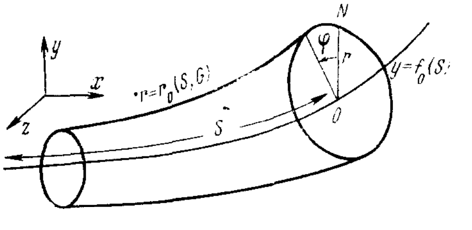

Fig. 1

We transform the equations of gasdynamics in the system of coordinates s, r and φ [5] by passing to new independent variables ψ and θ such that ψ = const and θ = const represent stream surfaces which can be introduced for analyzing three-dimensional stationary flows [6]. Using calculations similar to those in [7 - 9], we obtain for the determination of the seven dependent variables u, v, w, p, ρ, r and φ, as functions of independent variables s, ψ and θ, the following system consisting of five differential equations in partial derivatives and of two finite relationships:

$$\frac{\partial p}{\partial \psi} = \frac{1}{\partial \varphi / \partial \theta}\left[\frac{\partial \varphi}{\partial \psi}\frac{\partial p}{\partial \theta} - \gamma \frac{G(s, \theta, \psi)}{ur}\right] = \prod\nolimits_1 (s, \theta, \psi) \tag{1.1}$$

$$\frac{\partial r^2}{\partial \psi} = \frac{1}{\partial \varphi / \partial \theta}\left[\frac{2}{\rho u} - \frac{\partial \varphi}{\partial \psi}\frac{\partial r^2}{\partial \theta}\right] = \prod\nolimits_2 (s, \theta, \psi) \tag{1.2}$$

$$\frac{\partial w}{\partial s} = \pm \frac{u \sin \varphi}{R} - \frac{uv}{ru}\left(1 \mp \frac{r}{R}\cos \varphi\right) +$$

$$+ \frac{1 \mp r / R \cos \varphi}{ru \partial \varphi / \partial \theta}\left[\frac{1}{\gamma \rho}\frac{\partial p}{\partial \theta} + \frac{\partial r}{\partial \theta} G(s, \theta, \psi)\right] = \prod\nolimits_3 (s, \theta, \psi) \tag{1.3}$$

$$\frac{\partial r}{\partial s} = \frac{v}{\varkappa}\left(1 \mp \frac{r}{R}\cos \varphi\right) \tag{1.4}$$

$$\frac{\partial \varphi}{\partial s} = \frac{w}{ru}\left(1 \mp \frac{r}{R}\cos \varphi\right) = \prod\nolimits_4 (s, \theta, \psi) \tag{1.5}$$

$$p = \rho^{\gamma} \tag{1.6}$$

$$u = \left(\frac{\gamma + 1}{\gamma - 1} - 2\frac{p^{\gamma - 1/\gamma}}{\gamma - 1} - v^2 - w^2\right)^{1/2} \tag{1.7}$$

$$G(s, \theta, \psi) = \frac{u}{1 \mp r / R \cos \varphi}\frac{\partial v}{\partial s} \mp \frac{u^2 \cos \varphi}{R(1 \mp r / R \cos \varphi)} - \frac{w^2}{r}$$

Here γ is the adiabatic exponent; u, v and w are projections of the velocity vector on the axes of the curvilinear system of coordinates s, r and φ, respectively, normalized with respect to the critical speed of sound a_* ; p and ρ are respectively the pressure and the density of fluid, normalized with respect to pressure p_* and density ρ_* in the nozzle critical cross section; parameters with dimensions of length are normalized with respect to a certain characteristic length r_* and the stream-function with respect to $\rho_* a_* r_*^2$; $R(s)$ is the radius of curvature of curve $f_0(s)$.

For $w \equiv 0$ and $R = \infty$ the system (1.1) - (1.7) reduces in the general case to the system of equations which was used in [8] for numerical solving the inverse problem of the Laval nozzle of plane and axisymmetric configuration. The absence in Eq. (1.3) of derivatives with respect to ψ is important for the subsequent analysis.

In the general case of three-dimensional flow the inverse problem of the nozzle theory can be formulated for the system (1.1) - (1.7) as follows. Let the distribution of the velocity component $u = u_0(s, \theta)$ be specified in plane $r = r_0(s, \theta)$ and that of $w = w_0(\theta, \psi)$ and of coordinate $\varphi = \varphi_0(\theta, \psi)$ in plane $s = s_0$. We have to determine the family of stream surfaces and the flow parameters in the neighborhood of the reference stream surface.

The significant difference from the corresponding problem of plane and axisymmetric flows is that in this case it is not sufficient for solving the Cauchy problem to specify the stream data only at the stream surface, since the latter is a boundary layer surface. The two supplementary and the three compatibility equations [10] are in this case insufficient for the determination of flow parameters for the next layer (stream surface), since it is necessary to solve for the latter the system of Eqs. (1.3), (1.5) in partial derivatives for which boundary conditions are not stated.

Hence initial conditions have to be formulated not only at the stream surface but also at some surface different from the latter. Such statement of the inverse problem makes it possible to derive a unique solution and is, in a sense, equivalent to the statement of the inverse problem of plane vortex or axisymmetric flow, since for the latter it is also necessary to specify certain data (entropy distribution) in the reference plane. It is important to note that the specification of all velocity components at the stream surface results only in overdetermination.

2. Series expansion in terms of the streamfunction. Let us derive the solution of system (1.1) - (1.7) for the neighborhood of the reference surface $\psi = 0$ in the form of series expansion in terms of the streamfunction. Let us express the parameters u, v, w, p, ρ, φ and r in the form [8]

$$f(s, \psi, \theta) = \sum_{n=0}^{N} f_n(s, \theta)\psi^n + \sqrt{\psi}\sum_{n=0}^{N} f_n{}^0(s, \theta)\psi^n \qquad (2.1)$$

where $f(s, \psi, \theta)$ is any of the above functions. Substituting relationship (2.1) into the system (1.1) - (1.7) and equating the coefficients at like powers of ψ, for the determination of functions $f_n(s, \theta)$ and $f_n{}^\circ(s, \theta)$, we again obtain a system of equations in partial derivatives but with only two independent variables s and θ.

1. Derivation of solution in the case of $r_0(s, \theta) \neq 0$. For the determination of functions $f_0(s, \theta)$ we have the system (1.1) - (1.7) in which we denote all dependent variables by a zero subscript. Since all unknown functions, including w_0 and φ_0, are specified in the plane $s = s_0$, hence the system (1.3), (1.5) can be numerically integrated with respect to s over several planes $\theta = \text{const}$, and it is possible to determine functions $w_0(s, \theta)$ and $\varphi_0(s, \theta)$ throughout the reference plane $\psi = \text{const}$ and then determine functions p_0, ρ_0 and v_0 from relationships (1.4), (1.6) and (1.7).

We note that in an axisymmetric flow free of twist ($w = 0$, $R = \infty$) the problem is considerably simplified and it is possible to determine functions v_0, p_0 and ρ_0 directly from (1.4), (1.6) and (1.7). It is also important that for $u = u_0(s)$, $r = r_0(s)$ and $R(s) \neq \infty$, even if in the reference cross section $s = s_0$ functions w_0 and φ_0 vanish, they remain nonzero for any other s, since for $R \neq \infty$ system (1.3), (1.5) is nonhomogeneous. This means that a twist of the nozzle axis always results in the appearance of a peripheral velocity component.

It can be shown by simple but cumbersome calculations that the system of equations defining functions $f_0{}^\circ(s, \theta)$ is homogeneous with respect to these functions, while the specification of $w^\circ{}_0(s_0, \theta)$ and $\varphi_0{}^\circ(s_0, \theta)$ in the reference plane determines in the latter all remaining functions $f_0{}^\circ(s_0, \theta)$. Owing to the homogeneity of the related systems, it is obvious that all functions $f_0{}^\circ(s_0, \theta) \equiv 0$ when $w_0{}^\circ(s_0, \theta) = \varphi_0{}^\circ(s_0, \theta) \equiv 0$. The specification of condition $w_0{}^\circ(s_0, \theta) = \varphi_0(s_0, \theta) \equiv 0$ is physically justified, since it is difficult to visualize flows with infinite derivatives $\partial w / \partial\psi$ and $\partial\varphi / \partial\psi$ in regions away from the axis of symmetry of the reference plane.

Let us now consider the method of determining functions $f_1(s, \theta)$.

In the case of axisymmetric flows free of twist the pertinent linear equations are similar to those presented in [8], and the determination of function f_1 necessitates only the differentiation of the known functions $u_0(s)$ and $r_0(s)$. However in the case of a three-dimensional flow it is still necessary to solve the Cauchy problem for s and for the determination of functions w_1 and φ_1 to numerically integrate the system of equations concerned. In fact, it can be readily shown that any of the unknown functions can be represendted in the form

$$f_1(s,\theta)=\alpha_0(s,\theta)+\alpha_1(s,\theta)\,w_1+\alpha_2(s,\theta)\;\varphi_1$$

where $\alpha_i\,(s,\theta)$ are functions of known parameters $f_0\,(s,\theta)$ and their derivatives with respect to s and θ. The related system of linear equations becomes now a Cauchy system with the condition that $w_1\,(s_0,\theta)$ and $\varphi_1\,(s_0,\theta)$ are specified at the reference plane $s=s_0$. The validity of the proposed statement of the inverse problem for the three-dimensional case with $r_0\,(s,\theta)\not\equiv 0$ is thus proved.

2. Derivation of solution in the case of $r_0\,(s,\theta)\equiv 0$. For the determination of functions $f_0\,(s,\theta)$, $f_1\,(s,\theta)$ and $f_0{}^\circ\,(s,\theta)$ we have the following system of equations:

$$v_0{}^\circ=u_0\frac{\partial r_0{}^\circ}{\partial s},\qquad \frac{\partial\varphi_0}{\partial s}=\frac{w_0{}^\circ}{r_0{}^\circ u_0},\qquad r_0{}^\circ=\frac{2}{\rho_0u_0\partial\varphi_0/\partial\theta}\tag{2.2}$$

$$p_0{}^\circ=\mp\gamma\,\frac{\rho_0u_0{}^2r_0{}^2\cos\varphi_0}{R},\qquad p_1=-\frac{2}{r_0{}^\circ u_0\partial\varphi_0/\partial\theta}\times$$
$$\times\left\{\frac{\gamma}{2}\left[u_0\frac{\partial v_0{}^\circ}{\partial s}\mp u_0{}^2L_0{}^\circ R^{-1}+2u_0u_0{}^\circ\cos\varphi_0R^{-1}-(w_0{}^\circ)^{-2}\times\right.\right.$$
$$\left.\times(r_0{}^\circ)^{-1}\right]+r_0{}^0u_0\left(\frac{\partial\varphi_0}{\partial\theta}\right)^{-1}\left(p_0{}^\circ\frac{\partial\varphi_0{}^\circ}{\partial\theta}-\varphi_0{}^\circ\frac{\partial p_0{}^\circ}{\partial\theta}\right)+$$
$$\left.+\,p_0{}^\circ(r_0{}^\circ u_0{}^\circ+u_0r_1)\pm L_0{}^\circ r_0{}^\circ u_0p_0{}^\circ R^{-1}\right\}\tag{2.3}$$

$$r_1=-\frac{1}{3\partial\varphi_0/\partial\theta r_0{}^\circ}\left[\left(\frac{\rho_0{}^\circ}{\rho_0}+\frac{u_0{}^\circ}{u_0}\right)+(r_0{}^\circ)^2\frac{\partial\varphi_0{}^\circ}{\partial\theta}-\frac{1}{2}\frac{\partial\varphi_0{}^\circ}{\partial\theta}\frac{\partial(r_0{}^\circ)^2}{\partial\theta}\right]\tag{2.4}$$

$$\frac{\partial w_0{}^\circ}{\partial s}=\pm\left(\frac{u_0L^\circ{}_{01}}{R}+\frac{2u_0{}^\circ\sin\varphi_0}{R}\right)-\frac{w_0{}^\circ v_0{}^\circ}{u_0r_0{}^\circ}-$$
$$-\frac{1}{\gamma}\left\{\left[\frac{1}{2}p_0{}^\circ\frac{\partial r_1}{\partial\theta}-\frac{1}{2}r_0{}^\circ\frac{\partial p_1}{\partial\theta}+p_1\frac{\partial r_0{}^\circ}{\partial\theta}-r_1\frac{\partial p_0{}^\circ}{\partial\theta}\right]+\right.$$
$$\left.+\frac{1}{2}\left(p_0{}^\circ\frac{\partial r_0{}^\circ}{\partial\theta}-r_0{}^\circ\frac{\partial p_0{}^\circ}{\partial\theta}\right)\left(\frac{u_0{}^\circ}{u_0}\mp\frac{L_0{}^\circ}{R}\right)\right\}\tag{2.5}$$

$$\frac{\rho_0{}^\circ}{\rho_0}=\frac{1}{\gamma}\frac{p_0{}^\circ}{p_0},\qquad u_0{}^\circ=-\frac{p_0{}^\circ}{\gamma\rho_0u_0},\qquad L_0{}^\circ=r_0{}^\circ\cos\varphi_0$$
$$L_{01}{}^\circ=-\cos\varphi_0\;\varphi_0{}^\circ$$

We note that at $r_0{}^\circ\equiv 0$ from (2.2) follows that $w_0{}^\circ=v_0\equiv 0$. Final formulas for $f_0{}^\circ$, f_0, f_1 and $f_1{}^\circ$ in the case of axisymmetric flow are given in [8].

Let us first consider the case of $R=\infty$, for which from (2.2) and (2.5) we have $p_0{}^\circ=\rho_0{}^\circ=u_0{}^\circ\equiv 0$. It can be shown by simple, although cumbersome, calculations that for $R=\infty$ the system of equations defining functions $\varphi_0{}^\circ$, r_1, v_1 and w_1 is homogeneous, and for $\varphi_0{}^\circ(s_0,\theta)=w_1\,(s_0,\theta)\equiv 0$ yields the unique solution

$$\varphi_0{}^\circ\,(s,\theta)=w_1\,(s,\theta)=v_1\,(s,\theta)=r_1\,(s,\theta)\equiv 0$$

analogous to that for an axisymmetric flow free of twist.

The derivation of solution for functions $v_0{}^\circ$, $r_0{}^\circ$, p_1, φ_0 and $w_0{}^\circ$ for the determination of which we have Eqs. (2.2) - (2.5), proves to be far from trivial. It appears that Eq. (2.5) can be reduced to the form

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial w_0{}^\circ}{\partial s}\right)+\Phi_0(s,\theta)\,\frac{\partial}{\partial\theta}\left(\frac{\partial w_0{}^\circ}{\partial s}\right)+\Phi_1(s,\theta)\,\frac{\partial w_0{}^\circ}{\partial s}+\Phi_2(s,\theta)=0\tag{2.6}$$

where $\Phi_i\,(s,\theta)$ are known functions which can be calculated in the plane $s=\text{const}$, if $w_0{}^\circ$ and φ_0 are known and function $u_0\,(s)$ is specified. To find the derivative

$\partial w_0^\circ / \partial s$ on any plane $s = \text{const}$ it is necessary to solve numerically the boundary value problem for the ordinary differential equation (2.6). Owing to the periodicity of function $w_0^\circ(\theta)$ the boundary conditions for $\partial w_0^\circ / \partial s$ are that it must vanish in planes of symmetry. It is important to note that, when solving asymptotic equations for the vicinity of the reference surface, we determine not only the sought functions but also the derivative $\sigma\varphi / \sigma\psi$.

Solution of the nonlinear equations in partial derivatives derived in subsections 1 and 2 can generally be obtained only by numerical integration, although it is sometimes possible to obtain a solution in the closed form.

Let us derive an approximate solution of the system (2.2) – (2.5) for $K = \infty$ by the method of small perturbations. We represent functions φ_0, r_0°, v_0°, w_0° and p_1 in the form

$$\begin{aligned} \varphi_0 &= \theta + \varepsilon\varphi_{01}(s, \theta) + \varepsilon^2\varphi_{02}(s, \theta) + \ldots \\ r_0^\circ &= r_{00}^\circ(s) + \varepsilon r_{01}^\circ(s, \theta) + \varepsilon^2 r_{02}^\circ(s, \theta) + \ldots \\ w_0^\circ &= \varepsilon w_{01}^\circ(s, \theta) + \varepsilon^2 w_{02}^\circ(s, \theta) + \ldots \\ v_0^\circ &= v_{00}^\circ(s) + \varepsilon v_{01}^\circ(s, \theta) + \varepsilon^2 v_{02}^\circ(s, \theta) + \ldots \\ p_1 &= p_{10}(s) + \varepsilon p_{11}(s, \theta) + \varepsilon^2 p_{12}(s, \theta) + \ldots \end{aligned} \tag{2.7}$$

where ε is a certain small parameter. This form of presenting the unknown functions is based on the physically valid assumption that for the same distribution of velocity $u_0(s)$ along the axis the difference in the reference plane between the parameters of a three-dimensional and those of an axisymmetric flow is small. The substitution of relationships (2.7) into the system (2.2) – (2.5) and the linearization of the latter yields the following equations:

$$\begin{gathered} r_{00}^\circ(s) = (2 / \rho_0 u_0)^{1/2}, \qquad v_{00}^\circ(s) = u_0 dr_{00}^\circ(s) / ds \\ p_{10}(s) = -\frac{\gamma}{r_{00}^\circ(s)} \frac{dv_{00}^\circ}{ds} \end{gathered} \tag{2.8}$$

$$\begin{gathered} r_{00}^\circ \frac{\partial\varphi_{01}}{\partial\theta} = -2r_{01}^\circ, \quad v_{01} = u_0 \frac{\partial r_{01}^\circ}{\partial s}, \quad u_0 r_{00}^\circ \frac{\partial\varphi_{01}}{\partial s} = w_{01}^\circ \\ \frac{\partial}{\partial s}(r_{00}^\circ w_{01}^\circ) = \frac{1}{2}\left(\frac{\partial r_{01}}{\partial\theta}\frac{dv_{00}^\circ}{ds} - r_{00}^\circ \frac{\partial^2 v_{01}}{\partial s \partial\theta}\right) \end{gathered} \tag{2.9}$$

The system of equations defining the coefficients at ε^2 can be similarly derived. Formulas (2.8), which coincide with the corresponding formulas for an axisymmetric flow free of twist (see [8]), make it possible to determine functions $r_{00}(s)$, $v_{00}(s)$ and $p_{10}(s)$ by using finite relationships or by differentiating known functions with respect to s.

An important property of the system of Eqs. (2.9) in partial derivatives is their linearity which makes it possible to find the solution by the Fourier method of separation of variables. As the result we obtain to within terms of order ε^2 the following approximate solution of system (2.2) – (2.5):

$$r_0^\circ(s, \theta) = r_{00}(s)\left\{1 - \varepsilon r_{00}^\circ(s_0) \sum_{k=0}^{K} \frac{k}{2}\left[\frac{w_{01k}(s_0)}{2}\int_{s_0}^{s} \rho_0 ds + \frac{\varphi_{01k}(s_0)}{r_{00}^0(s_0)}\right]\cos k\theta\right\}$$

$$\varphi_0(s, \theta) = \theta + \varepsilon \sum_{k=0}^{K}\left[\frac{r_{00}^0(s_0)\, w_{01k}^\circ(s_0)}{2}\int_{s_0}^{s} \rho_0 ds + \varphi_{01k}(s_0)\right]\sin k\theta$$

$$p_1(s, \theta) = p_{10}\left\{1 - \varepsilon r_{00}^\circ(s_0) \sum_{k=0}^{K} k\left[\frac{w_{01k}^\circ(s_0)}{2}\int_{s_0}^{s} \rho_0 ds + \frac{\varphi_{01k}(s_0)}{r_{00}^\circ(s_0)}\right]\cos k\theta\right\}$$

$$v_0^\circ(s,\theta)=v_{00}^\circ(s)-\varepsilon\frac{r_{00}^\circ(s_0)}{2}\sum_{k=0}^{K}k\left\{v_{00}^\circ(s)\left[\frac{\overset{\circ}{w}_{10k}(s_0)}{2}\int_{s_0}^{s}\rho_0ds+\right.\right.$$

$$\left.\left.+\frac{\varphi_{01k}(s_0)}{r_{00}^\circ(s_0)}\right]+\frac{\overset{\circ}{w}_{01k}(s_0)}{r_{00}^\circ(s_0)}\right\}\cos k\theta \tag{2.10}$$

$$w_0^\circ(s,\theta)=\varepsilon r_{00}^\circ(s_0)\,[r_{00}^\circ(s)]^{-1}\sum_{k=0}^{K}\overset{\circ}{w}_{01k}(s_0)\sin k\theta$$

where $\overset{\circ}{w}_{01k}(s_0)$ and $\varphi_{01k}(s_0)$ are the Fourier coefficients in the expansions of input functions $w_0^\circ(s_0,\theta)$ and $\varphi_0(s_0,\theta)$ in terms of $\sin k\theta$. It follows from relationships (2.10) that the shape of the cross section can be altered by varying at $s=s_0$ the input data $\overset{\circ}{w}_{01k}(s_0)$ and $\varphi_{01k}(s_0)$. These relationships also imply that, if at $s=s_0$ we set $\varphi_{01k}(s_0)=0$, the nozzle cross section is in that plane circular, while with increasing s it loses its axial symmetry and assumes the form determined by the input value of $w_{01k}(s_0)$. Unlike in the axisymmetric case, $\partial r_0^\circ/\partial s\,|_{s=s_0}$ is now a function of θ. If all $w_{01k}^\circ(s_0)\equiv 0$, except $\overset{\circ}{w}_{012}(s_0)$, then obviously a three-dimensional flow has two planes of symmetry and for $\overset{\circ}{w}_{011}(s_0)\neq 0$ it has one such plane. Formulas (2.10) are evidently valid for calculating subsonic, as well as mixed transonic flows. A similar solution can be derived in the case of $r_0\equiv 0$ and $R\neq\infty$ using, for example, $1/R$ for ε. It can be shown that in this case a nozzle of circular cross section in the reference plane remains circular to within ε^2 in all other cross sections. Its axis is, however, curvilinear in accordance with the law $R=R(s)$.

3. Difference scheme for solving the inverse problem of the Laval nozzle theory and examples of nozzles of noncircular cross sections. The derivation of this scheme for numerically solving the inverse problem of three-dimensional flow is analogous to that used in [8] for the axisymmetric case. The implicit three-point scheme, proposed there, of second-order accuracy and with a variable pitch on layers $\psi=$ const, ensures stable solutions in the subsonic region, where the Cauchy problem for elliptic equations is generally improper, as well as in the transonic and supersonic regions. Below we present only the extension of this scheme to three-dimensional flows without specific proof, which can be found in [8].

We denote layers $\psi=$ const and planes $\theta=$ const and $s=$ const by subscripts n, j and l respectively. Let us assume that all parameters $u, v, w, p, \rho, r, \varphi$ and $\partial\varphi/\partial\psi$ are known in the layer $\psi_n=$ const and in the planes L and J defined, respectively, by $s=$ const and $\theta=$ const. The L , as well as the J-planes are generally not equispaced. The calculation of parameters in the layer $\psi_{n+1}=$ const is carried out by the method of iteration in the ψ- and s-directions, as follows. First, using formulas

$$p_{jl\,(n+1)}^{(\nu)}=p_{jln}+\frac{\Delta\psi}{2}\,[\Pi_{1jln}+\Pi_{1jl\,(n+1)}^{(\nu-1)}] \tag{3.1}$$

$$r_{jl\,(n+1)}^{(\nu)}=\left\{r_{jln}^2+\frac{\Delta\psi}{2}\,[\Pi_{2jln}+\Pi_{2jl\,(n+1)}^{(\nu-1)}]\right\}^{1/2} \tag{3.2}$$

$$v_{jl\,(n+1)}^{(\nu)}=\left(\frac{\partial r}{\partial s}\right)_{jl\,(n+1)}^{(\nu)}\left\{\frac{\gamma+1}{\gamma-1}-\frac{2\,[p_{jl\,(n+1)}]^{\gamma-1/\gamma}}{\gamma-1}-\right.$$

$$\left.-[w_{jl\,(n+1)}^{(\nu-1)}]^2\right\}^{1/2}\left\{\left[\left(\frac{\partial r}{\partial s}\right)_{jl\,(n+1)}^{(\nu)}\right]^2+\left[\left(1\mp\frac{r}{R}\cos\varphi\right)_{jl\,(n+1)}^{(\nu-1)}\right]^2\right\}^{-1/2} \tag{3.3}$$

we successively determine in each iteration the magnitudes $r^{(\nu)}_{jl(n+1)}$, $p^{(\nu)}_{jl(n+1)}$ and $v^{(\nu)}_{jl(n+1)}$. In these equations ν is the iteration number and $\Delta\psi$ is the pitch of the difference scheme in the ψ-direction. In the first iteration ($\nu = 1$) functions with superscript ($\nu - 1$) are assumed to be equal to the corresponding functions in the nth layer, and in all subsequent iterations the parameters derived in the preceding iterations are used.

Then, using the known in the $(n+1)$-st layer parameters $p^{(\nu)}_{jl(n+1)}$, $r^{(\nu)}_{jl(n+1)}$ and $v^{(\nu)}_{jl(n+1)}$, starting from the plane $s = s_0$ we integrate in the s-direction over all J-planes (except planes of symmetry) the system of Eqs. (1.3) and (1.5) whose right-hand sides now depend only on the unknown functions w and φ. Parameters $w^{(\nu i)}_{j(l+1)(n+1)}$ and $\varphi^{(\nu i)}_{j(l+1)(n+1)}$ are determined in each of the j-planes at the $(n+1)$-st layer ψ_{n+1} and in the plane $s_{(l+1)}$ respectively, by formulas

$$w^{(\nu i)}_{j\,(l+1)\,(n+1)} = w^{(\nu)}_{jl\,(n+1)} + \frac{\Delta s}{2}\{\Pi^{(\nu)}_{3jl\,(n+1)} + \Pi^{[\nu\,(i-1)]}_{3j\,(l+1)\,(n+1)}\} \tag{3.4}$$

$$\varphi^{(\nu i)}_{j\,(l+1)\,(n+1)} = \varphi^{(\nu)}_{jl\,(n+1)} + \frac{\Delta s}{2}\{\Pi^{(\nu)}_{4jl\,(n+1)} + \Pi^{[\nu\,(i-1)]}_{4j\,(l+1)\,(n+1)}\} \tag{3.5}$$

where i is the iteration number and Δs is the pitch of the difference scheme in the s-direction. In the first iteration ($i = 1$) functions with superscript $[\nu\,(i-1)]$ are assumed to be equal to the corresponding functions in the preceding s_l-plane, and the parameters derived in the preceding iteration are used in all subsequent iterations.

Integration of the system (1.3), (1.5) in the s-direction is carried out from the reference plane $s = s_0$, in which functions $w(\psi, \theta)$ and $\varphi(\psi, \theta)$ are specified, up to a certain end-plane $s = s_k$. Having determined functions p, r, w, v and φ in the νth iteration, from formulas (1.6) and (1.7) we find u and ρ. If the difference of values of all unknown functions obtained in the νth iteration and those in the $(\nu - 1)$-st iteration is outside the required accuracy, the $(\nu + 1)$-st iteration is carried out. In the $(\nu + 1)$-st iteration we again use formulas (3.1) – (3.3) for calculating functions p, r and v in the $(n+1)$-st layer and then numerically integrate system (1.3), (1.5) (formulas (3.4) and (3.5)) with respect to s and determine functions w and φ. The ν- and i- iterations are completed on reaching the required accuracy. This is followed by calculations of the next layer $\psi = \text{const}$.

It is expedient to calculate derivatives $\partial r/\partial s$, $\partial v/\partial s$, $\partial p/\partial\theta$, $\partial r/\partial\theta$ and $\partial\varphi/\partial\theta$ along layers $\psi = \text{const}$ and $\theta = \text{const}$ by the three-point scheme with varying pitch proposed in [8]. This ensures a stable solution of the Cauchy problem in the elliptic region. The proposed scheme is of second-order accuracy in all directions.

The examination of system (1.1) – (1.7) shows the possibility of applying it in the calculation of the practically important twisting flows by using the difference scheme proposed in [8]. In fact, by assuming that all flow parameters (except φ) are functions of only two independent variables s and ψ, and that $R = \infty$, from (1.3) we readily obtain

$$w(s, \psi) = \frac{w_0(s, \psi)\, r_0(s_0, \psi)}{r(s, \psi)} \tag{3.6}$$

where $w_0(s_0, \psi)$ and $r_0(s_0, \psi)$ are the values of functions w and r in the reference plane. From Eq. (1.5) then follows that

$$\varphi = \theta + \int_{s_0}^{s} \frac{w(s, \psi)}{ru}\, ds \tag{3.7}$$

and, consequently, that $\partial\varphi/\partial\theta = 1$. The form of Eqs. (1.1), (1.2) and (1.4) is now exactly the same as in the axisymmetric case, except that the expression for $G(s, \psi)$ has the term w^2/r added to it which, in accordance with (3.6), is a known function of ψ, and that w^2 is to be taken into consideration in the expression for u. The difference scheme described in [8] is used for calculating such flow.

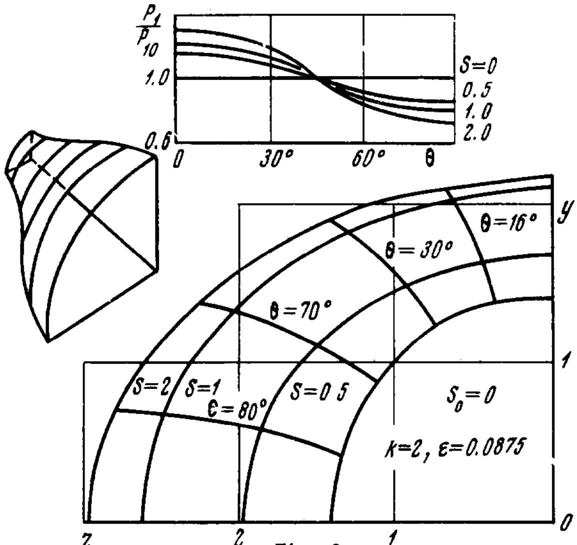

Fig. 2.

In conclusion we examine some examples of calculating sub- and supersonic flows in nozzles and channels of varying cross sections. First, let us consider an axisymmetric flow free of twist. The proposed here difference scheme was used in [11] for investigating the flow in annular nozzles of various shapes.

Three-dimensional flows were calculated exclusively by the asymptotic formulas (2.10) which provide a qualitative appraisal of nozzle geometry and of the distribution of parameters. Results of these calculations are shown in Figs. 2 and 3.

In Fig. 2 is shown a nozzle which up to its minimum cross section is axisymmetric and close to elliptic in two planes downstream of it. These cross sections have been drawn for $s = \text{const}$. At the outlet plane $s = 2$ the ratio of the cross section semiaxes is equal 1.5. In the calculation $s_0 = 0$, $k = 2$, $\varepsilon = 0.0875$, $\gamma = 1.4$, $r_0 = 0$ and $R = \infty$ were assumed, and u_0 was defined by formula

$$u_0(s) = 1 + \frac{(1 - u_\infty)(\bar{u}_\infty - 1)(e^{-s/b} - 1)}{(1 - u_\infty)e^{-s/b} + (\bar{u}_\infty - 1)} \tag{3.8}$$

where $u_\infty = 0.1$, $\bar{u}_\infty = 1.9$ and $1/b = 3.5$. In this Figure is also shown the locus of points $\theta = \text{const}$, i.e., the projections on the yz-plane of secondary stream surfaces and the ratio p_1/p_{10} at various cross sections in terms of θ. The presence of gas leakage from plane $z = 0$ to plane $y = 0$ is noticeable.

The results of calculations of a nozzle which is axisymmetric up to its minimum cross section has a single plane of symmetry, and a noncircular outlet cross section downstream are similarly presented in Fig. 3. It shows, as previously, the shape of various cross sections at $\theta = \text{const}$. The axisymmetric nozzle geometry for the same initial distribution is shown there by dotted lines. In calculating this variant it was assumed that $s_0 = 0$,

$k = 1$, $\varepsilon = 0.175$, $\gamma = 1.4$, $r_0 = 0$ and $R(s) = \infty$, and $u_0(s)$ was determined by formula (3.8). In this case gas leakage occurred from plane $\varphi = 0$ to plane $\varphi = 180°$ It is significant that the center of gravity of cross section shifts with increasing s to the left

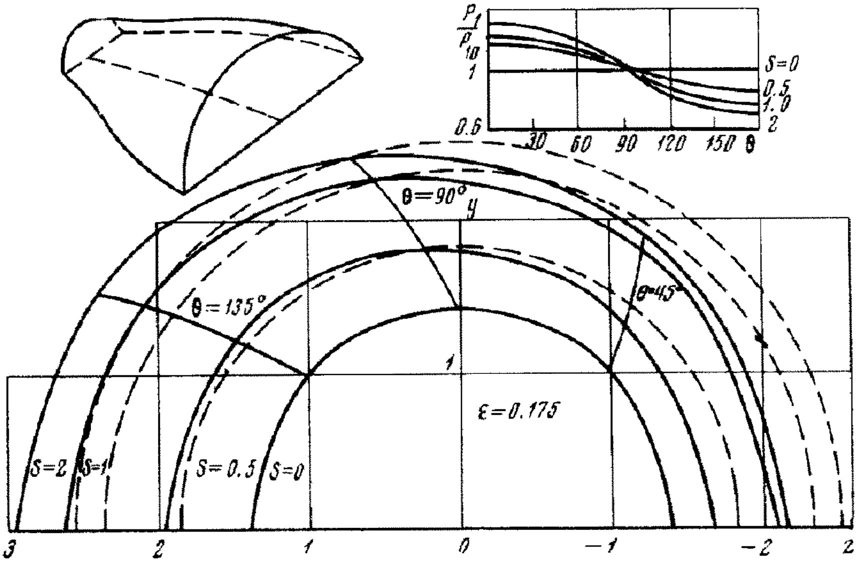

Fig. 3

of its position in the critical cross section. This results in a twist of the nozzle geometric axis and the appearance of a side force.

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FREE OSCILLATIONS OF LIQUID IN RIGID VESSELS

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Small steady oscillations of a perfect incompressible liquid in a rigid vessel are examined. Although this problem was fairly thoroughly investigated [1 - 3], the determination of high frequency oscillations and of their form in a liquid in vessels of arbitrary shape presents considerable difficulties.

A simplified approximate method, whose accuracy increases at higher frequencies is proposed for solving this problem. It is shown on the example of several problems that for practical purposes this method can be used for the full range of frequencies. Estimates of the lower and upper bounds are given in some of the problems.

1. The velocity potential Φ of free oscillations of liquid satisfies the Laplace equation with boundary conditions [4]

$$\Delta\Phi = 0 \quad \text{in } V$$

$$\frac{\partial\Phi}{\partial z} - \frac{\lambda^2}{R}\Phi = 0 \quad \left(\lambda^2 = \frac{\omega^2 R}{g}\right) \text{ along } \Sigma, \quad \frac{\partial\Phi}{\partial n} = 0 \text{ along } S \tag{1.1}$$

where S is the wetted part of the vessel surface, Σ is the free surface of the unperturbed liquid. V is the region bounded by the surface $S + \Sigma$, $\partial\Phi / \partial n$ is a derivative along the normal to S, R is a constant of dimension length (a characteristic dimension of the cavity), ω is the angular oscillation frequency, g is the acceleration of gravity, and the direction of the Oz -axis is opposite to that of the gravity force vector (Fig. 1).

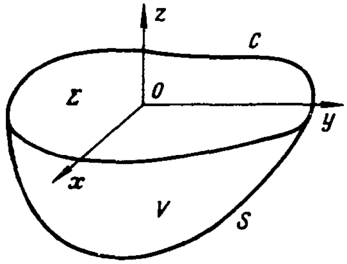

Fig. 1

Let us establish a certain property of function Φ for $\lambda \to \infty$. Assuming that functions in Green's formula are equal to Φ, with the use of (1.1) we obtain

$$\iiint\limits_V \left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] dV = \frac{R}{\lambda^2}\iint\limits_\Sigma \left(\frac{\partial\Phi}{\partial z}\right)^2 d\Sigma$$

Hence